A KOBAYASHI METRIC VERSION OF BUN WONG'S THEOREM

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1. Introduction

1.1. **Basic Terminology and Statement of Main Theorem.** For a complex manifold M, denote by k_M its Kobayashi-Royden infinitesimal metric and by d_M its Kobayashi distance, and (see [KRA1], [KOB2]).

Definition 1.1. A map $f : M \to N$ from a complex manifold M into another complex manifold N is said to be a *Kobayashi isometry* if f is a homeomorphism satisfying the condition that $d_N(f(x), f(y)) = d_M(x, y)$ for every $x, y \in M$.

The set G_M of Kobayashi isometries of M (onto M itself) endowed with the compact-open topology is a topological group with respect to the binary law of composition of mappings. We call this group G_M, the *Kobayashi isometry group* of the complex manifold M.

Denote by $\mathbb{B}^n$ the open unit ball in $\mathbb{C}^n$. Notice that the Kobayashi distance of $\mathbb{B}^n$ in fact coincides with the Poincaré-Bergman distance of $\mathbb{B}^n$. The primary aim of this article is to establish the following theorem, which is a Kobayashi metric version of Bun Wong's classical theorem [WON].

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{C}^n$ with a $\mathcal{C}^{2,\epsilon}$ smooth ($\epsilon > 0$), strongly pseudoconvex boundary. If its Kobayashi isometry group G_Ω is non-compact, then Ω is biholomorphic to the open unit ball $\mathbb{B}^n$.*

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Remark 1.1. The classical results (see [WON] and [ROS]) assume noncompact (biholomorphic) automorphism group of the domain Ω. But it has been understood for many years that this theorem of Bun Wong and Rosay is really a "flattening" result of geometry (terminology of Gromov is being used here). Our new theorem puts this relationship into perspective. $\square$

1.2. **The Kobayashi-distance version of Wong's theorem.** Experts who are familiar with Wong's theorem [WON] would expect the following:

Theorem 1.2 (Seshadri-Verma). *Let Ω be a bounded domain in $\mathbb{C}^n$ with a $\mathcal{C}^2$ smooth, strongly pseudoconvex boundary. If its Kobayashi isometry group G_Ω is non-compact, then there exists a Kobayashi isometry $f : \Omega \to \mathbb{B}^n$.*

Certainly it was Seshadri and Verma ([SEV1, SEV2] as well as [VER]) who first conceived the idea of a metric version of Wong's theorem. We present here a different proof of Theorem 1.2 which closes some gaps in the extant argument and answers some subtle questions. We believe that the arguments we present in this article can be of use for many other purposes. The arguments involved here are subtle, because the mappings under consideration are *a priori* only continuous. Unlike the holomorphic case the restrictions of Kobayashi isometries to sub-domains are not isometries with respect to the Kobayashi metric of the sub-domain. Another slippery point is that the full power of Montel's theorem and Cartan's uniqueness theorem is not available for equi-continuous maps. So it is necessary to give precise estimates and arguments that clarify all these subtle points necessary for the proof. It is true that our proof of this theorem follows the same general line of reasoning as [SEV1, SEV2], which in turn is along the line of scaling method introduced by S. Pinchuk around 1980 ([PIN]).

The second half, that is indeed the main part of this article, presents the following:

Theorem 1.3. *Let Ω be a bounded domain in $\mathbb{C}^n$ with a $\mathcal{C}^{2,\epsilon}$ smooth, strongly pseudoconvex boundary. If there exists a Kobayashi isometry $f : \Omega \to \mathbb{B}^n$ from Ω onto the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$, then f is either holomorphic or conjugate holomorphic.*

Notice that Theorems 1.2 and 1.3 imply Theorem 1.1. And that is the main point of this paper. The work [SEV2], which addresses similar questions, assumes that the mapping is $\mathcal{C}^1$ to the boundary. We are able to eliminate this somewhat indelicate hypothesis.

Seshadri and Verma in the above cited work have proved the same conclusion in the case when Ω is strongly convex. Notice that our theorem here assumes only *strong pseudoconvexity.* On the other hand, the experts in this line of research would feel that the optimal regularity of the boundary should be $\mathcal{C}^2$, instead of $\mathcal{C}^{2,\epsilon}$ with some $\epsilon > 0$. At the time of this writing, we do not know how to achieve the optimum, because of technical reasons connected with work of Lempert [LEM1, LEM2]. We would like to mention it as a question for future study.

2. Some Fundamentals

2.1. **Terminology and Notation.** Let Ω be a Kobayashi hyperbolic domain in $\mathbb{C}^n$. For a point $q \in \Omega$, let us write

$$G_\Omega(q) = \{\varphi(q) \mid \varphi \in G_\Omega\}.$$

This is usually called the (point) *orbit* of q under the action of the Kobayashi isometry group G_Ω on the domain Ω.

Call a boundary point $p \in \partial\Omega$ a *boundary orbit accumulation point* if p belongs to the closure $\overline{G_\Omega(q)}$ of the orbit $G_\Omega(q)$ of a certain interior point $q \in \Omega$ under the action of the Kobayashi isometry group G_Ω. In other words, p is a boundary orbit accumulation point if and only if there exists an interior point $q \in \Omega$ and a sequence of Kobayashi isometries $\varphi_j \in G_\Omega$ such that $\lim_{j\to\infty} \varphi_j(q) = p$.

Let us adopt the notation

$$B_d(q; r) := \{y \mid d(y, q) < r\}$$

for any distance d in general. Then one observes the following:

Proposition 2.1. *Let Ω be a bounded, complete Kobayashi hyperbolic domain in $\mathbb{C}^n$. Then its Kobayashi isometry group G_Ω is non-compact if and only if Ω admits a boundary orbit accumulation point.*

Proof. Notice that the *sufficiency* is obvious. We establish the *necessity* only.

Expecting a contradiction, assume to the contrary that there are no boundary orbit accumulation points. Then, for every point q of the domain Ω, the orbit of q under the group action is relatively compact. Now let $\{\varphi_j\}$ be an arbitrarily chosen sequence of Kobayashi isometries; then it is obviously an equi-continuous family with respect to the Kobayashi distance. By Barth's theorem ([BAR]), this implies that φ_j forms an equi-continuous family on compact subsets with respect to the Euclidean distance. Thus one may use the Arzela-Ascoli theorem

to extract a sequence $\{\varphi_{j_k}\}$ that converges uniformly on compact subsets of Ω to a limit mapping $\widehat{\varphi}$. Thus, replacing φ_j by a subsequence, one may assume without loss of generality that φ_j converges uniformly on compacta to a continuous map, say $\widehat{\varphi}$.

Since the point orbit is always compact, $\widehat{\varphi}(a) = b$ for some $a, b \in \Omega$. Notice that, exploiting the completeness of d_Ω, one can deduce that

$$\widehat{\varphi}(\Omega) = \widehat{\varphi}\Big(\bigcup_{\nu=1}^{\infty} B_{d_\Omega}(a;\nu)\Big) = \bigcup_{\nu=1}^{\infty} \widehat{\varphi}(B_{d_\Omega}(a;\nu)) \subset \bigcup_{\nu=1}^{\infty} B_{d_\Omega}(b;\nu) = \Omega.$$

It is obvious that one can apply the same argument to the sequence φ_j^{-1} (replacing it by a subsequence that converges uniformly on compacta, if necessary). Thus $\widehat{\varphi} : \Omega \to \Omega$ is a homeomorphism. It is obvious that $\widehat{\varphi}$ preserves the Kobayashi distance d_Ω. Altogether, it follows that $\widehat{\varphi} \in G_\Omega$. This establishes that G_Ω is compact. (This argument shows the sequential compactness of G_Ω, to be precise. But then G_Ω equipped with the topology of uniform convergence on compacta is metrizable.) This contradiction yields the desired conclusion. □

Since every bounded strongly pseudoconvex domain is complete with respect to the Kobayashi distance (see [GRA]), Theorem 1.2 now follows by the following more general statement:

Theorem 2.1. *Let Ω be a domain in $\mathbb{C}^n$. If Ω admits a boundary orbit accumulation point at which the boundary of Ω is $\mathcal{C}^2$ smooth, strongly pseudoconvex, then Ω is Kobayashi isometric to the open unit ball in $\mathbb{C}^n$.*

Notice that for this theorem one does not need to assume that the domain has to be *a priori* bounded or Kobayashi hyperbolic. Instead, the complete Kobayashi hyperbolicity of the domain will be obtained along the way in the proof, from the given hypothesis only. The proof of this result will be developed in Section 3.

2.2. **Some Comparison Estimates.** We now give some comparison inequalities for the Kobayashi distances and the Kobayashi-Royden infinitesimal metrics for a sub-domain against its ambient domain. This will play a crucial role in the proof of Theorem 2.1.

Lemma 2.1. (Kim-Ma) *Let Ω be a Kobayashi hyperbolic domain in $\mathbb{C}^n$ with a subdomain $\Omega' \subset \Omega$. Let $q, x \in \Omega'$, let $d_\Omega(q, x) = a$, and let $b > a$. If Ω' satisfies the condition $B_{d_\Omega}(q; b) \subset \Omega'$, then the following*

two inequalities hold:

$$d_{\Omega'}(q,x) \leq \frac{1}{\tanh(b-a)}\ d_\Omega(q,x),$$

$$k_{\Omega'}(x,v) \leq \frac{1}{\tanh(b-a)}\ k_\Omega(x,v),\ \ v \in \mathbb{C}^n.$$

[Recall here that k is the infinitesimal Kobayashi/Royden metric and d is the integrated Kobayashi/Royden distance.]

Proof. For the sake of the reader's convenience, we include here the proof, lifting it from [KIMA]. Let $s = \tanh(b-a)$ and let $\epsilon > 0$. Denote by Δ the open unit disc in $\mathbb{C}$ and by $\Delta(a;r)$ the open disc of Euclidean radius r centered at a in $\mathbb{C}$. Then, by definition of $k_\Omega(x,v)$, there exists a holomorphic map $h : \Delta \to \Omega$ such that $h(0) = x$ and $h'(0) = v/(k_\Omega(x,v)+\epsilon)$. If $\zeta \in \Delta(0;s)$, then

$$\begin{aligned} d_\Omega(q,h(\zeta)) &\leq d_\Omega(q,x) + d_\Omega(x,h(\zeta)) \\ &= a + d_\Omega(h(0),h(\zeta)) \\ &\leq a + d_\Delta(0,\zeta) \\ &< a + (b-a) \\ &= b. \end{aligned}$$

This shows that $h(\Delta(0;s)) \subset \Omega'$. Now define $g : \Delta \to \Omega'$ by $g(z) := h(sz)$. Then one has $g(0) = x$ and $g'(0) = sh'(0) = sv/(k_\Omega(x,v)+\epsilon)$. This implies that $k_{\Omega'}(x,v) \leq (k_\Omega(x,v)+\epsilon)/s$. Since ϵ is an arbitrarily chosen positive number, it follows that

$$k_{\Omega'}(x,v) \leq \frac{1}{\tanh(b-a)}\ k_\Omega(x,v), \quad \forall v \in \mathbb{C}^n.$$

Now let δ be chosen such that $0 < \delta < b-a$. There is a $\mathcal{C}^1$ curve $\gamma : [0,1] \to \Omega$ such that $\gamma(0) = q$, $\gamma(1) = x$, and $\int_0^1 k_\Omega(\gamma(t),\gamma'(t))\ dt < a+\delta$. This implies that $d_\Omega(q,\gamma(t)) < a+\delta < b$ for any $t \in [0,1]$. Hence $\gamma(t) \in \Omega'$ for every $t \in [0,1]$. Notice that the inequality

$$k_{\Omega'}(\gamma(t),\gamma'(t)) \leq k_\Omega(\gamma(t),\gamma'(t))/\tanh(b-a-\delta)$$

holds for every $t \in [0,1]$ by the preceding arguments. But then this implies that

$$\begin{aligned} d_{\Omega'}(x,q) &\leq \int_0^1 k_{\Omega'}(\gamma(t),\gamma'(t))\ dt \\ &\qquad\qquad \leq \frac{1}{\tanh(b-a-\delta)} \int_0^1 k_\Omega(\gamma(t),\gamma'(t)\ dt. \end{aligned}$$

Consequently, one deduces that $d_{\Omega'}(x,q) < (a+\delta)/\tanh(b-a-\delta)$. Now, letting δ tend to 0, one obtains the desired conclusion. $\square$

3. Scaling With a Sequence of Kobayashi Isometries; Proof of Theorem 2.1

Now we present a precise and detailed proof of Theorem 2.1.

Denote by $\mathbb{B}(p;r) = \{z \in \mathbb{C}^n \mid |z-p| < r\}$, the open ball of radius r centered at p with respect to the Euclidean distance on $\mathbb{C}^n$.

Because of the $\mathcal{C}^2$ strong pseudoconvexity of $\partial\Omega$ at the boundary orbit accumulation point p, there exist a positive real number ε and a biholomorphic mapping $\Psi : U \to \mathbb{B}(0;\varepsilon)$ such that $\Psi(p) = 0$,

$$\Psi(\Omega \cap U) = \{(z_1,\ldots,z_n) \in \mathbb{B}(0;\varepsilon) \mid \\ \operatorname{Re} z_1 > |z_1|^2 + \ldots + |z_n|^2 + E(z_1,\ldots,z_n)\ \},$$

and

$$\Psi(\partial\Omega \cap U) = \{(z_1,\ldots,z_n) \in \mathbb{B}(0;\varepsilon) \mid \\ \operatorname{Re} z_1 = |z_1|^2 + \ldots + |z_n|^2 + E(z_1,\ldots,z_n)\ \},$$

where

$$E(z_1,\ldots,z_n) = o(|z_1|^2 + \ldots + |z_n|^2).$$

Apply now the localization method by N. Sibony that uses only the plurisubharmonic peak functions. (See [SIB], [BER], [GAU], [BYGK] for instance, as well as [ROY].) It follows from the hypothesis that every open neighborhood U of p in $\mathbb{C}^n$ admits an open set V in $\mathbb{C}^n$ such that $p \in V \subset\subset U$ and

$$\frac{1}{2}\, k_{\Omega\cap U}(z,\xi) \le k_\Omega(z,\xi)$$

for every $z \in \Omega \cap V$ and every ξ in the tangent space $T_z\Omega$ $(= \mathbb{C}^n)$ of Ω at the point $z \in \Omega$. It can also be arranged that

$$\frac{1}{2}\, d_{\Omega\cap U}(x,y) \le d_\Omega(x,y)$$

for every $x, y \in \Omega \cap V$. See [BYGK] for instance for this last inequality. This in particular implies

> The localization property: *For any open neighborhood W of p and for any relatively compact subset K of Ω, there exists a positive integer j_0 such that $\varphi_j(K) \subset \Omega \cap W$ whenever $j > j_0$.*

Notice that one can take W such that $\Omega \cap W$ equipped with the Kobayashi distance $d_{\Omega\cap W}$ is Cauchy complete. Consequently the localization property, together with the fact that p is a boundary orbit accumulation point, implies that Ω is complete Kobayashi hyperbolic.

Next we apply Pinchuk's scaling method [PIN]. Let

$$\Psi \circ \varphi_j(q) \equiv (q_{1j}, \ldots, q_{nj})$$

for $j = 1, 2, \ldots$. Fix j for a moment. Choose $p_{1j} \in \mathbb{C}$ such that

$$(p_{1j}, q_{2j}, \ldots, q_{nj}) \in \Psi(\partial\Omega \cap U)$$

and

$$q_{1j} - p_{1j} > 0.$$

Let $\zeta = A_j(z)$ for the complex affine map $A_j : \mathbb{C}^n \to \mathbb{C}^n$ defined by

$$\begin{aligned}
\zeta_1 &= e^{i\theta_j}(z_1 - p_{1j}) + \sum_{k=2}^{n} c_{kj}(z_k - q_{kj}) \\
\zeta_2 &= z_2 - q_{2j} \\
&\vdots \\
\zeta_n &= z_n - q_{nj},
\end{aligned}$$

where the real number θ_j and the complex numbers $c_{2j}, \ldots, c_{nj}$ are chosen so that the real hypersurface $A_j \circ \Psi(\partial\Omega \cap U)$ is tangent to the real hyperplane defined by the equation $\mathrm{Re}\,\zeta_1 = 0$. It is important to notice now, for the computation in the later part of this proof, that

$$\lim_{j\to\infty} e^{i\theta_j} = 1 \quad \text{and} \quad \lim_{j\to\infty} c_{mj} = 0$$

for every $m \in \{2, \ldots, n\}$.

Then define $\Lambda_j : \mathbb{C}^n \to \mathbb{C}^n$ by

$$\Lambda_j(z_1, \ldots, z_n) = \left(\frac{z_1}{\lambda_j}, \frac{z_2}{\sqrt{\lambda_j}}, \ldots, \frac{z_n}{\sqrt{\lambda_j}} \right),$$

where $\lambda_j = q_{1j} - p_{1j}$.

Exploit now the multi-variable Cayley transform

$$\Phi(z_1, \ldots, z_n) = \left(\frac{1-z_1}{1+z_1}, \frac{2z_2}{1+z_1}, \ldots, \frac{2z_n}{1+z_1} \right),$$

and consider the following sequences

$$\sigma_j := \Phi \circ \Lambda_j \circ A_j \circ \Psi|_U ,$$

and

$$\tau_j := \sigma_j \circ \varphi_j ,$$

for $j = 1, 2, \ldots$.

Notice that each σ_j maps U into $\mathbb{C}^n$. It plays the role of a holomorphic embedding of $\Omega \cap U$ into $\mathbb{C}^n$. On the other hand, the domain of definition of τ_j has to be considered more carefully. Thanks to the *localization property* above, for every compact subset K of Ω, there exists a positive integer $j(K, U)$ such that τ_j maps K into $\mathbb{C}^n$ for every $j \geq j(K, U)$. Thus, for each such K, one is allowed to consider $\tau_j|_K : K \to \mathbb{C}^n$ *only for the indices* j *with* $j \geq j(K, U)$.

Now a direct calculation shows that, shrinking U if necessary, for every $\epsilon > 0$ there exists a positive integer N such that

$$\mathbb{B}(0; 1-\epsilon) \subset \sigma_j(\Omega \cap W) \subset \mathbb{B}(0; 1+\epsilon)$$

for every $j > N$. Thus, replacing N by $N + j(K, U)$, one may conclude that

$$\tau_j(K) \subset \mathbb{B}(0; 1+\epsilon)$$

for every $j > N$.

Take now a sequence $\{K_\nu\}$ of relatively compact subsets of Ω satisfying the following three conditions:

(i) K_ν is a relatively compact, open subset of Ω for each ν;
(ii) $\overline{K_\nu} \subset K_{\nu+1}$, for $\nu = 1, 2, \ldots$;
(iii) $\displaystyle\bigcup_{\nu=1}^{\infty} K_\nu = \Omega$.

Such a sequence $\{K_\nu\}$ is usually called a *(relatively) compact exhaustion sequence* of Ω.

Given a relatively compact exhaustion sequence $\{K_\nu\}$ of Ω, we consider the restricted sequences $\{\tau_{j,\nu} = \tau_j|_{K_\nu} \mid j = 1, 2, \ldots\}$, for every $\nu = 1, 2, \ldots$.

Using these restricted sequences, we would like to establish:

Claim (†): *There exists a compact exhaustion sequence $\{K_\nu\}$ such that the sequence τ_j admits a subsequence that converges uniformly on every compact subset of Ω to a Kobayashi isometry $\widehat{\tau} : \Omega \to \mathbb{B}$ from the domain Ω onto the ball $\mathbb{B}$.*

Notice that the proof of Theorem 2.1 is complete as soon as this Claim is established.

Apply now Lemma 2.1 to the domain Ω. Let ν be an arbitrarily chosen positive integer. Let $z_0 \in \Omega$. Let the Kobayashi metric ball $B_{d_\Omega}(z_0; \mu\nu)$ play the role of the subdomain Ω', where μ is an integer with $\mu > 5$. Then, for any $x, y \in B_{d_\Omega}(z_0, \nu)$, it holds that

$$d_{B_{d_\Omega}(z;2\mu\nu)}(x, y) \leq \frac{1}{\tanh(\mu\nu)}\, d_\Omega(x, y).$$

Exploiting the fact that (Ω, d_Ω) is Cauchy complete, we now choose the relatively compact exhaustion sequence consisting of expanding Kobayashi metric balls:

$$K_\nu \equiv B_{d_\Omega}(q; \nu).$$

Take $N > 0$ such that $\varphi_j(q) \in V \cap \Omega$ and $\varphi_j(K_{2\mu\nu}) \subset \Omega \cap U$ whenever $j > N$. Enlarging N if necessary, we may achieve also that $\sigma_j(\Omega \cap U) \subset \mathbb{B}(0; 1 + \epsilon)$ for every $j > N$. Moreover, for any $x, y \in K_\nu$, one sees that:

$$\begin{aligned}
d_{\mathbb{B}(0;1+\epsilon)}(\tau_j(x), \tau_j(y)) &\leq d_{\sigma_j(\Omega\cap U)}(\sigma_j \circ \varphi_j(x), \sigma_j \circ \varphi_j(y)) \\
&= d_{\Omega\cap U}(\varphi_j(x), \varphi_j(y)) \\
&\leq d_{B_{d_\Omega}(\varphi_j(q),;2\mu\nu))}(\varphi_j(x), \varphi_j(y)) \\
&\leq \frac{1}{\tanh(\mu\nu)}\, d_\Omega(\varphi_j(x), \varphi_j(y)) \\
&= \frac{1}{\tanh(\mu\nu)}\, d_\Omega(x, y).
\end{aligned}$$

As a summary, we have that

$$d_{\mathbb{B}(0,1+\epsilon)}(\tau_j(x), \tau_j(y)) \leq \frac{1}{\tanh(\mu\nu)}\, d_\Omega(x, y), \quad \forall x, y \in K_\nu. \tag{1}$$

This estimate shows that the sequence $\{\tau_j\}$ is an equi-continuous family on each K_ν. Therefore one may extract a subsequence that

converges uniformly on every compact subset of Ω to a continuous map $\widehat{\tau} : \Omega \to \overline{\mathbb{B}(0; 1+\epsilon)}$.

Now the above estimate yields

$$d_{\mathbb{B}(0;1+\epsilon)}(\widehat{\tau}(x), \widehat{\tau}(y)) \leq \frac{1}{\tanh(\mu\nu)}\, d_\Omega(x,y), \quad \forall x, y \in K_\nu. \tag{2}$$

Since this estimate must hold for every $\epsilon > 0$, one deduces first that $\widehat{\tau}(K_\nu) \subset \overline{\mathbb{B}(0;1)} = \overline{\mathbb{B}}$. But then, using the distance estimate above one sees immediately that $\tau_j(K_\nu)$ for any j is bounded away from the boundary of $\mathbb{B}$. So $\widehat{\tau}(K_\nu) \subset \mathbb{B}$ for every ν. Consequently $\widehat{\tau}$ maps Ω into $\mathbb{B}$.

Moreover,

$$d_{\mathbb{B}}(\widehat{\tau}(x), \widehat{\tau}(y)) \leq \frac{1}{\tanh(\mu\nu)}\, d_\Omega(x,y)), \qquad \forall x, y \in K_\nu.$$

Letting $\mu \to \infty$, this last estimate turns into

$$d_{\mathbb{B}}(\widehat{\tau}(x), \widehat{\tau}(y)) \leq d_\Omega(x,y)), \qquad \forall x, y \in K_\nu. \tag{3}$$

Now let $x, y \in \Omega$ be fixed. Choose $0 < \delta < 1/2$ such that $\widehat{\tau}(x), \widehat{\tau}(y) \in \mathbb{B}(0; 1-2\delta)$. Then there exists a positive integer N_1 such that

$$\tau_j(x), \tau_j(y) \in \mathbb{B}(0; 1-\delta)$$

and

$$\mathbb{B}(0; 1-\delta) \subset \sigma_j(\Omega \cap U)$$

whenever $j > N_1$. Thus one obtains

$$\begin{aligned} d_\Omega(x,y) &= d_\Omega(\varphi_j(x), \varphi_j(y)) \\ &\leq d_{\Omega\cap U}(\varphi_j(x), \varphi_j(y)) \\ &\leq d_{\sigma_j(\Omega\cap U)}(\sigma_j \circ \varphi_j(x), \sigma_j \circ \varphi_j(y)) \\ &= d_{\sigma_j(\Omega\cap U)}(\tau_j(x), \tau_j(y)) \\ &\leq d_{\mathbb{B}(0;1-\delta)}(\tau_j(x), \tau_j(y)). \end{aligned}$$

Let j tend to infinity first, and then let δ converge to zero. Then one deduces that

$$d_\Omega(x,y) \leq d_{\mathbb{B}}(\widehat{\tau}(x), \widehat{\tau}(y)). \tag{4}$$

Combining (3) and (4), one sees that

$$d_\Omega(x,y) = d_{\mathbb{B}}(\widehat{\tau}(x), \widehat{\tau}(y)).$$

Since x and y have been arbitrarily chosen points of Ω, it follows that $\widehat{\tau} : \Omega \to \mathbb{B}$ preserves the Kobayashi distance.

In order to complete the proof of the claim and hence Theorem 2.1, it remains to show that $\widehat{\tau} : \Omega \to \mathbb{B}$ is surjective. Let $y \in \mathbb{B}$. Then there exists r with $0 < r < 1$ such that $|y| < r$. Moreover, there exists $N_2 > 0$ such that $\tau_j^{-1}(y) \in \Omega$ for every $j > N_2$.

Let $x_j = \tau_j^{-1}(y)$. Then it holds that $d_\Omega(q, x_j) \leq d_{\mathbb{B}}(0; y) + 1$ for every $j > N_2$. Therefore a subsequence x_{j_k} of x_j converges, say, to $\widehat{x} \in \Omega$. Now, because of the uniform convergence of τ_j to $\widehat{\tau}$ on compacta, one immediately sees that

$$\widehat{\tau}(\widehat{x}) = \widehat{\tau}(\lim_{j\to\infty}(x_j)) = [\lim_{k\to\infty} \tau_k](\lim_{j\to\infty}(x_j)) = \lim_{j\to\infty} \tau_j(x_j) = y.$$

This shows that $\widehat{\tau} : \Omega \to \mathbb{B}$ is surjective. Consequently, the proof of Claim (†) follows. The proof of Theorem 2.1 is now complete. □

4. Complex Analyticity of the Kobayashi Isometry $f : \Omega \to \mathbb{B}^n$

It now remains to establish Theorem 1.3. Incidentally, it seems appropriate for us to pose the following naturally arising question:

Question 4.1. *Let n be a positive integer. Let Ω_1 and Ω_2 be bounded domains in $\mathbb{C}^n$ with $\mathcal{C}^2$ smooth, strongly pseudoconvex boundaries, and let $f : \Omega_1 \to \Omega_2$ be a homeomorphism that is an isometry with respect to the Kobayashi distances. Then, is f or $\overline{f}$ necessarily holomorphic?*

We do not know the answer to this question at present; we show in this paper that the answer is affirmative in case $\Omega_2 = \mathbb{B}^n$ and $\partial\Omega_1$ is $\mathcal{C}^{2,\epsilon}$ smooth.

4.1. **Burns-Krantz construction of Lempert discs for strongly pseudoconvex domains.**

Here we would like to explain how Burns and Krantz adapted Lempert's analysis to the strongly pseudoconvex domains, as this set of ideas is going to play an important role for our proof. In what follows, let $\Omega \subseteq \mathbb{C}^n$ be a bounded, strongly pseudoconvex domain with $\mathcal{C}^{2,\epsilon}$ boundary.

Let $p \in \partial\Omega$. Then by Burns and Krantz [BUK], there exist open neighborhoods V and U of p such that $p \in V \subset\subset U$ such that any $p' \in \partial\Omega \cap V$ admits a Lempert disc $\varphi : \overline{\Delta} \to \overline{\Omega}$ such that $p' \in \varphi(\partial\Delta)$ and $\varphi(\overline{\Delta}) \subset U$. Perhaps the term Lempert disc needs to be clarified. In fact we will do more than that. We will quickly describe what Burns and Krantz present in Proposition 4.3 and Lemma 4.4 of [BUK].

Take the Fornaess embedding ([FOR]) that embeds Ω holomorphically and properly into a strongly convex domain $\Omega' \subset \mathbb{C}^N$ with some $N > n$. This embedding map, say F, is in fact smooth up to the

boundary of Ω such that $F : \overline{\Omega} \to \overline{\Omega'}$ is smooth, and also $F(\partial\Omega) \subset \partial\Omega'$. Let $T_{F(p)}(F(\Omega))$ be the tangent plane to $F(\Omega)$ at $F(p)$. This is an n-dimensional complex affine space in $\mathbb{C}^N$ and so we may identify it with the standard $\mathbb{C}^n$. Let $\Pi : \mathbb{C}^N \to T_{F(p)}(F(\Omega))$ be the orthogonal projection. Then $\Pi \circ F : \Omega \to \mathbb{C}^n$ is a holomorphic mapping, and furthermore is a injective holomorphic mapping of $\Omega \cap U$ for some open neighborhood U of p in $\mathbb{C}^n$.

Denote by $F' := \Pi \circ F$. Since $F'(\Omega)$ is bounded, there exists a sufficiently large ball B such that $\Omega \subset B$ and $p \in \partial\Omega \cap \partial B$. Slide B slightly along the inward normal direction of $\partial F'(\Omega)$ at $F'(p)$ and call it B', so that the point p is now outside B' and yet all the rest of $F'(\Omega)$ is within B' except a small neighborhood U' of p satisfying $U' \subset U$. Namely $F'(\Omega) \subset B' \cup U'$ with $U' \subset U$.

Consider the convex hull the union of $F'(\Omega)$ and B' and call it Ω'. This domain is convex and its boundary near $F(p)$ coincides the boundary of $F'(\Omega)$. Note that the boundary of Ω' is neither smooth nor strictly convex. However it is easy to modify Ω' slightly so that the newly modified domain Ω'' is strongly convex with $\mathcal{C}^6$ boundary, and yet the boundary of Ω'' coincides with the boundary of $F'(\Omega)$ in a neighborhood of $F'(p)$.

Then one considers Lempert discs, i.e., the holomorphic discs that are isometric-and-geodesic embeddings of Δ, for the domain Ω''. (See [LEM2] for this.) If one considers the Lempert discs centered at a point sufficiently close to p with the direction at p nearly complex parallel to a complex tangent direction to the boundary of Ω'' at p, then the image of such discs will be within a small neighborhood of p. Moreover such Lempert discs, say h, are also holomorphic geodesic embeddings of the disc Δ into Ω, in the sense that $\check{h} := [F']^{-1} \circ h$ is the holomorphic geodesic embedding of the unit disc Δ into Ω. For further details, see the above cited text in [BUK], especially Proposition 4.3 and Lemma 4.4 therein.

For convenience, we shall call these discs *Lempert-Burns-Krantz discs* for the strongly pseudoconvex domain Ω, or *LBK-discs* for short. Such discs exist at a point sufficiently close to the boundary along the directions that are approximately complex tangential to the boundary.

4.2. **Holomorphicity along the Lempert-Burns-Krantz discs.** Take now an LBK-disc $h : \Delta \to \Omega$ in the domain Ω such that $h^* d_\Omega = d_\Delta$. Then we first present:

Proposition 4.1. *For any continuous Kobayashi distance isometry $f : \Omega \to \mathbb{B}^n$, the composition $f \circ h : \Delta \to \mathbb{B}^n$ is holomorphic or conjugate-holomorphic.*

Proof. Denote by $\widetilde{h} := f \circ h$. We give the proof in two steps.

Step 1: *The mapping $\widetilde{h}$ is C^∞ smooth.* We shall first show that $\widetilde{h}$ is smooth at the origin. Take three points a, b and c in the unit disc such that the Poincaré geodesic triangle, say $T(a, b, c)$, with vertices at these three points contains the origin in its interior. Fill $T(a, b, c)$ with the geodesics from a to points on the geodesic joining b and c. Then obviously the origin is on one of these geodesics. Now, let m denote the foot of this geodesic. This procedure defines a smooth diffeomorphism, say h, from a Euclidean triangle onto $T(a, b, c)$, having two parameters: one is the time parameter of each geodesic from the point a to a point on the geodesic joining b and c, and the other is the parameter of the geodesic joining b and c.

Let $\widetilde{a} = \widetilde{h}(a), \widetilde{b} = \widetilde{h}(b)$ and $\widetilde{c} = \widetilde{h}(c)$. Let $\widetilde{T}(\widetilde{a}, \widetilde{b}, \widetilde{c})$ be the geodesic triangle in $\mathbb{B}^n$ with respect to the Poincaré metric, with vertices at the three points $\widetilde{a}, \widetilde{b}$ and $\widetilde{c}$. Again one may fill this triangle by Poincaré geodesics of the ball, namely by the geodesics joining $\widetilde{a}$ to the points on the geodesic joining $\widetilde{b}$ and $\widetilde{c}$. This will again yield a smooth diffeomorphism from a Euclidean triangle onto the filled triangle $\widetilde{T}(\widetilde{a}, \widetilde{b}, \widetilde{c})$. Since the Kobayashi distance-balls are strongly convex for both d_Δ and $d_{\mathbb{B}^n}$, it follows that $\widetilde{T}(\widetilde{a}, \widetilde{b}, \widetilde{c}) = \widetilde{h}(T(a, b, c))$. Moreover $\tilde{h}$ maps each geodesic to a corresponding geodesic with matching speed. This shows that $\widetilde{f}$ is indeed smooth at the origin. As the argument can be easily modified to prove the smoothness of $\widetilde{h}$ at any point of the disc Δ, the map $\widetilde{h} : \Delta \to \mathbb{B}$ is $\mathcal{C}^\infty$ smooth at every point.

Acknowledgement: Notice that this argument can be used to give a proof of the well-known theorem of Myers-Steenrod ([MYS]). This simple but elegant and powerful technique was conveyed to the authors by Robert E. Greene in a private communication. We acknowledge with a great pleasure our indebtedness to him.

Step 2. *The mapping $\widetilde{h}$ is holomorphic or conjugate holomorphic.* Since $\widetilde{h}$ maps geodesics to geodesics, it is a geodesic embedding. Thus the surface $\widetilde{T}(\widetilde{a}, \widetilde{b}, \widetilde{c})$ has the maximal holomorphic sectional curvature, and this can be realized only by holomorphic sections in the ball. (Notice that the Kobayashi metric coincides with the Poincaré metric in the unit ball, and hence it is Kähler with negative constant holomorphic

sectional curvature.) Thus the tangent plane to the surface $\widetilde{T}(\widetilde{a}, \widetilde{b}, \widetilde{c})$ is complex. Since $d\widetilde{h}_*(T_*\Delta)$ is always a complex subspace in $T_{\widetilde{h}(*)}\mathbb{B}^n$, it follows by a standard argument that $\widetilde{h}$ is either holomorphic or conjugate holomorphic. □

4.3. **The Lempert map for strongly pseudoconvex domains.** Recall that our domain Ω'' is a bounded strongly convex domain with $\mathcal{C}^6$ boundary. Let $x \in \Omega''$. Then, for an arbitrarily chosen $z \in \Omega''$ with $z \neq x$, there exists a unique Lempert disc $h_{x,z} : \Delta \to$ such that $h_{x,z}(0) = x$ and $h_{x,z}(\lambda) = z$ for some λ with $0 < \lambda < 1$. Then in [LEM1, LEM2] Lempert defines $\Phi(z) = \lambda {h_{x,z}}'(0)$ and shows that $\Phi : \Omega'' \to \mathbb{C}^n$ extends to a $\mathcal{C}^2$ smooth map of the closure $\overline{\Omega''}$. We shall call x the *pivot* of the map Φ.
(Although Lempert was mainly interested in the representation mapping $\widetilde{\Phi}(z) = \lambda {h_{x,z}}'(0)/|{h_{x,z}}'(0)|$, which is nowadays known as the Lempert representation map $\widetilde{\Phi} : \overline{\Omega''} \to \overline{\mathbb{B}^n}$, we remark here that both Φ and $\widetilde{\Phi}$ are known to be $\mathcal{C}^2$ smooth up to the boundary ([LEM2]).)

In the next subsection, we will see how to use this map Φ to show that the Kobayashi isometry $f : \Omega \to \mathbb{B}^n$ in question is $\mathcal{C}^2$ up to the boundary.

4.4. **Smooth extension of Kobayashi isometry to the boundary.** Choose $p' \in S$ that is sufficiently close to p. Then let $h : \Delta \to \Omega''$ be an LBK-disc with $h(0) = F'(p')$, as mentioned above. Let us continue to use the notation $\check{h} := [F']^{-1} \circ h$. Then consider the Möbius transformation $\mu : \mathbb{B}^n \to \mathbb{B}^n$ which maps $f(p')$ to the origin. The preceding arguments imply that the composition $\widehat{h} := \mu \circ f \circ \check{h} : \Delta \to \mathbb{B}^n$ defines a Lempert disc at the origin for the unit ball $\mathbb{B}^n$. Thus it is linear. Moreover, it follows immediately that $\widehat{h}(\lambda) = \lambda \widehat{h}'(0) = \lambda(\mu \circ f \circ \check{h})'(0) = d[\mu \circ f]_{p'}(\lambda \check{h}'(0))$.

It is known that our f, a Kobayashi distance isometry, admits a Lipschitz 1/2 extension ([HEN]). But with the strong assumption that Ω is bounded strongly pseudoconvex with $\mathcal{C}^6$ boundary, we shall prove the following:

Proposition 4.2. *The Kobayashi isometry $f : \Omega \to \mathbb{B}$ has a $\mathcal{C}^2$ extension to the boundary. More precisely, there exists an open neighborhood W of $\partial\Omega$ in $\mathbb{C}^n$ such that $f : \overline{\Omega} \cap W \to \overline{\mathbb{B}}$ is $\mathcal{C}^2$ smooth.*

Proof. Notice first that the Kobayashi isometry f as well as f^{-1} are locally Lipschitz with respect to the Euclidean metric, as the Kobayashi

distance generates the same topology as the Euclidean distance ([BAR]). Therefore the set

$$S := \{x \in \Omega \mid df_x \text{ and } d[f^{-1}]_{f(x)} \text{ exist } \}$$

is a subset of full measure, i.e., the Lebesgue measure of S is the same as the Lebesgue measure of Ω. In particular, S is dense in Ω.

Now let $\Upsilon(h(\lambda)) := [d[\mu \circ f]_{p'}]^{-1} \circ \mu \circ f(\check{h}(\lambda))$ for $\lambda \in \Delta$. Since $\mu \circ f \circ \check{h}(\lambda) = d[\mu \circ f]_{p'}(\lambda \check{h}'(0))$, it follows that Υ coincides, at every point on the image $h(\Delta)$, with the aforementioned map $\Phi : \Omega'' \to \mathbb{C}^n$ with its pivot at p'. To be precise for any $\zeta \in h(\Delta)$, let $h(\lambda) = \zeta$. Then $\Upsilon(\zeta) = \lambda h'(0)$.

Altogether, one sees that the mapping $[d[\mu \circ f]_p]^{-1} \circ \mu \circ f \circ [F']^{-1}$ coincides with Υ in a small conical neighborhood (with apex at p) of $h(\partial\Delta)$ filled by the LBK-discs at p'; it follows that f is $\mathcal{C}^2$ smooth in an open neighborhood of $\check{h}(\partial\Delta)$. Now it is easy to observe that this gives rise to the $\mathcal{C}^2$ smoothness of f in an open neighborhood of $\partial\Omega$ as desired. □

4.5. **The Kobayashi isometry is CR on the boundary.** We now present

Proposition 4.3. *The restriction of the extension of f to $\partial\Omega$ into $\mathbb{C}^n$ is a CR function (or an anti-CR function).*

Proof. Let $p \in \partial\Omega$ and let $\overline{L} \in T_p^{0,1}\partial\Omega$. Regard this vector field as a derivation operator on the Euclidean space $\mathbb{C}^n$. Then, for every $\epsilon > 0$, there exists an $r \in (0, \epsilon)$ and $q \in \Omega$ with $|p - q| = r$ such that we may find a Lempert disc $\varphi : \Delta \to \Omega$ satisfying

$$\overline{L}_q f = \frac{\partial}{\partial \bar{\zeta}}\Big|_0 f \circ \varphi.$$

Since $f \circ \varphi$ is holomorphic (replace it by $\bar{f} \circ \varphi$ if necessary), one immediately sees that $\overline{L}_q f = 0$ Since f is $\mathcal{C}^2$ up to the boundary, letting r tend to zero, one obtains the assertion. □

4.6. **Analyticity of the Kobayashi isometry – proof of Theorem 1.3.** Finally we are ready to present:

Proof of Theorem 1.3. Start with Proposition 4.3 which we just proved. Recall that f restricted to $\partial\Omega$ is a C^2 smooth CR map from $\partial\Omega$ to $\partial\mathbb{B}$.

It is also a local diffeomorphism. Notice that $f(\partial\Omega)$ is compact and relatively open. Therefore $f(\partial\Omega) = \partial\mathbb{B}$. This implies that f is a covering map. However $\partial\Omega$ is simply connected if $n > 1$, being topologically a sphere. Hence $f : \partial\Omega \to \mathbb{B}$ is a C^2 diffeomorphism.

Now apply the Bochner-Hartogs theorem. The mapping f extends to a holomorphic mapping, say $\widehat{f}$ of Ω into $\mathbb{B}$. Now restrict f to a Lempert-Burns-Krantz disc. This has the same value as $\widehat{f}$ at any point of the disc. Therefore, f and $\widehat{f}$ must coincide at every point of the LBK-disc. Altogether, the map f itself is holomorphic in $W \cap \Omega$ for some open neighborhood W of $\partial\Omega$.

Then one may ask whether $f = \widehat{f}$ on Ω. They do coincide indeed. This can be seen as follows. Since $f : \Omega \to \mathbb{B}$ is a Kobayashi distance isometry, Ω itself has the property that any two points in it must have one and the only one shortest distance realizing curve joining them. Take any such curve $\gamma : [0, \ell] \to \Omega$ with $\gamma(0), \gamma(\ell) \in W \cap \Omega$. Then one observes the following:

- $d_\Omega(\gamma(s), \gamma(t)) = s - t,$ whenever $0 \le t \le s \le \ell$.
- $f \circ \gamma$ is the unique distance realizing curve joining $f(\gamma(0))$ and $f(\gamma(\ell))$.
- $f(\gamma(0)) = \widehat{f}(\gamma(0))$ and $f(\gamma(\ell)) = \widehat{f}(\gamma(\ell))$.

Now for every $t \in [0, \ell]$ notice that

$$d_\Omega(f \circ \gamma(0), \widehat{f} \circ \gamma(t)) = d_\Omega(\widehat{f} \circ \gamma(0), \widehat{f} \circ \gamma(t)) \le d_\Omega(\gamma(0), \gamma(t)) = t,$$

and likewise

$$d_\Omega(\widehat{f} \circ \gamma(t), f \circ \gamma(\ell)) \le \ell - t.$$

Now by triangle inequality and the uniqueness of the (shortest) distance realizing curve joining two points, one sees immidiately that the inequalities above are equalities and that

$$\widehat{f}(\gamma(t)) = f(\gamma(t))$$

for every $t \in [0, \ell]$. It is now immediately deduced that $f = \widehat{f}$ on Ω. In particular f (and hence $\widehat{f}$) is a bijective holomorphic mapping of Ω onto the ball $\mathbb{B}$. This finishes the proof. □

Note: The authors would like to thank H. Seshadri for asking whether f can be shown directly to coincide with $\widehat{f}$; we clarified it changing the end of the proof slightly.

5. Concluding Remarks

The original theorem of Bun Wong [WON], and variants by Rosay [ROS] and others, has proved to embody a powerful and far-reaching set of ideas. In particular, it was consideration of this insight that led Greene and Krantz ([GRK2], [GRK3], [GRK4]) to formulate the principle that the Levi geometry of a boundary orbit accumulation point will determine the global geometry of the domain. This in turn has led to the Greene-Krantz conjecture: that a boundary orbit accumulation point for a smoothly bounded domain must in fact be of finite type in the sense of Kohn-Catlin-D'Angelo.

The result of these studies has been a profound and fruitful development in geometric analysis. We wish that the present contribution will lead to further insights.

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